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LETTER TO THE EDITOR

A solvable model of quantum discontinuities

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Abstract. It is shown that an appropriate triple-well version of the binding potential $V(x)=ax^2+bx^4+cx^6$ $(a, c>0,\ b<0,\ b^2>3ac)$ provides a solvable model whereby the discontinuous behaviour of the energy eigenvalues of any finite set of lowest-lying levels can be deduced analytically. The reason for the occurence of such discontinuities will be discussed.

Singular perturbations, even in infinitesimal amounts, can lead to remarkable consequences. They can do so by creating energetically favourable subsidiary potential wells. Following an observation due to Herbst and Simon [1] this fact was brought to the fore by Calogero [2]. Varma and co-workers [3] provided further numerical confirmation.

Using model Hamiltonians, Calogero showed that the energy eigenvalues in such cases are not necessarily continuous functions of the perturbing couplings. As the perturbation is gradually made to vanish, the energy eigenvalues do not tend to those of the unperturbed problem. Consequently, when the relevant couplings are ultimately switched off from their infinitesimal values, the energies jump abruptly to conform to the dictums of the unperturbed Hamiltonian. The system is said to display quantum discontinuities.

Calogero notes further that the occurence of discontinuities is always accompanied by a loss of normalizability by the associated wavefunctions.

Moreover, in all his models, the limiting process forces the subsidiary wells to grow infinitely wide. Their minima recede to infinity and their depths either remain fixed or diverge. Hence, prior to the discontinuous jump, there is a tendency amongst a group of levels to accumulate near the minimum of the relevant subsidiary well.

In short, under the limiting process, one witnesses a discontinuous behaviour of the energies, a loss of normalizability by the wavefunctions and a marked convergence of levels near a suitable minimum. Analytically, the discontinuity of only a single eigenvalue per potential is deducible in the examples of Calogero.

We propose a solvable and very instructive model to illustrate the phenomenon of discontinuities. In this model the discontinuous behaviour of any finite set of lowest-lying levels can be deduced analytically. The question of the apparent loss of normalizability will be discussed. No accumulation of levels *à la* Calogero materializes.

Our model is a suitable version of the one-dimensional sextic anharmonic potential $V(x)=ax^2+bx^4+cx^6$. The considerations to be presented can be extended straightforwardly to higher dimensions. Turbiner [4] showed that if the couplings a, b, c $(c>0)$ are appropriately tuned, the Hamiltonian develops an intimate connection with an underlying SL(2, R) symmetry. A subset of lowest-lying levels then splits off from the rest and becomes tractable by elementary means. The utility of such solutions has been discussed at length in a recent paper [5]. In this work, we follow the approach of [5] but adequate details will be provided so as to keep the work self-contained. The historical development of such exact solutions is also outlined in this last mentioned reference.

Of immediate interest to us is the case $a, c > 0$ and $b < 0$ such that $b^2 > 3ac$ and $b, c \to 0$ in a manner to be stipulated shortly. The potential has a triple-well structure under these conditions.

Using oscillator scales the Schrödinger equation in dimensionless variables is

$$\frac{d^2\psi}{dx^2} + [\varepsilon - x^2 + \lambda x^4 - \beta^2 x^6]\Psi = 0. \tag{1}$$

Here λ is positive, β is taken positive by convention and the potential is $V(x) = x^2 - \lambda x^4 + \beta^2 x^6$. Make the following substitutions in (1):

$$\Psi = v(x)\exp\left(\frac{\gamma x^2}{2} - \frac{\beta x^4}{4}\right) \qquad v = \sum_{n=0} a_n x^n \tag{2}$$

$$\lambda = 2\beta\gamma \tag{3}$$

with $a_0 = 1$, $a_1 = 0$ for even parity solutions and $a_0 = 0$, $a_1 = 1$ for odd parity solutions. Instead of (1) we now have the three-term relation

$$(n+3)(n+4)a_{n+4} + [\varepsilon + \gamma(2n+5)]a_{n+2} + [\gamma^2 - 1 - \beta(2n+3)]a_n = 0. \tag{4}$$

This permits $v(x)$ to be a polynomial of degree k provided

$$a_k \neq 0 \qquad a_{k+2} = a_{k+4} = 0. \tag{5}$$

Conditions (5) require that

$$\gamma^2 = 1 + \beta(2k+3). \tag{6}$$

The corresponding energy eigenvalues are the roots of the equation $a_{k+2} = 0$. These can most easily be obtained by expressing a_{k+2} as an $(m+1) \times (m+1)$ determinant as explained in [5]. The index m equals $k/2$ or $(k-1)/2$ depending on whether k is even or odd. There are always $(m+1)$ real roots due to the underlying SL(2, R) symmetry [4, 5]. For k even (odd) these represent the lowest $(m+1)$ eigenvalues for even (odd) levels. A given set of solutions (k fixed) corresponds to a given potential since both β and λ can be held fixed. The wavefunctions follow through a repeated use of the recursion relation (4) along with the appropriate parity condition.

To facilitate the discussion in what follows we explicitly list the first few results.

(i) $k = 0$: We recover only the ground state solution with

$$\varepsilon_0 = -\gamma \qquad \gamma = \sqrt{1+3\beta} \qquad v_0 = 1 \qquad \lambda = 2\beta\gamma. \tag{7}$$

(ii) $k = 1$: We have the one-node solution with

$$\varepsilon_1 = -3\gamma \qquad \gamma = \sqrt{1+5\beta} \qquad v_1 = x, \qquad \lambda = 2\beta\gamma. \tag{8}$$

(iii) $k = 2$: We get the zero and two-node solutions. We shall need them only in the $\beta \to 0$ limit. Accordingly, they can be approximated as

$$\varepsilon_0 \approx -5 - \frac{39\beta}{2} \qquad v_0 \approx 1 + 4x^2 \qquad \gamma \approx 1 + \frac{7\beta}{2} \tag{9}$$

and

$$\varepsilon_0 \approx -1 - \frac{3\beta}{2} \qquad v_2 \approx 1 - \beta x^2 \qquad \gamma \approx \frac{1+7\beta}{2}. \tag{10}$$

(iv) For a general k, we obtain $(m+1)$ eigenvalues. Ignoring $O(\beta)$ terms the sequence

$$-(2k+1), -(2k-3), -(2k-7), \ldots, -1 \text{ or } -3 \tag{11}$$

results, according as k is even or odd.

The discontinuous behaviour of these energy eigenvalues follows straightforwardly. Let $\beta, \lambda \to 0$ in accord with (3) and (6). The potential $V(x) \to x^2$ whose energy eigenvalues $\varepsilon_n = 2n+1$, $n = 0, 1, 2, \ldots$, do not emerge as the limiting values. A discontinuous jump must result as β and λ are switched off from their infinitesimal values.

As $\beta \to 0$, at first glance, wavefunctions seem to lose the property of normalizability. For example, the wavefunction Ψ_0 corresponding to v_0 of (7) tends to $\exp(+x^2/2)$. A little later we shall return to this question and discuss it in detail.

An amusing fact emerges under the limiting process. All the even (odd) unnormalized wavefunctions tend to suitable nodeless (one-node) functions as $\beta, \lambda \to 0$. The receding wells carry the nodes away with them to $\pm\infty$. As an example, consider the two-node function for $k = 2$. The nodes are at $|x| = 1/\beta^{1/2}$ and recede to $\pm\infty$ as $\beta, \lambda \to 0$. In the same limit $\Psi_2(k=2) \to \Psi_0(k=0)$. Such a change of nodal characteristics has not been noted previously [2, 3].

However, there is no accumulation of levels accompanying the limiting process. There is, of course, an expected tendency amongst successive even and odd levels to become pairwise degenerate. To see this, observe that (for small β) once a negative level appears, its energy from then on is, to an excellent approximation, a linear function of k. For example, for $k = 1, 3, 5, 7, \ldots$ ε_1 is almost at $-3, -7, -11, -15, \ldots$ respectively. Hence for $k = 2, 4, 6, \ldots$ it should almost be at $-5, -9, -13, \ldots$ repectively, precisely where ε_0 is for such k values.

The reason for the lack of accumulation of levels can be easily understood by paying attention to the geometry of the triple-well potential in the limit $\beta, \lambda \to 0$ with (3) and (6) satisfied. In this limit both the depth and the width of the outer wells are finite and independent of β and λ. The depth below the $E = 0$ line grows as $(2k+3)$ and the width at the $E = 0$ line grows as $(2k+3)^{1/2}$ approximately. For fixed k, they remain fixed and finite. Hence no accumulation of levels occurs. For large k the outer wells are deep so that pairwise degeneracy of even and odd levels occurs as it does for any symmetric deep double well.

The present model has another distinctive feature. Here, an oscillator-like spectrum goes over into an oscillator spectrum despite a discontinuous jump and the accompanying drastic change in the wavefunctions.

We now turn to a very illuminating aspect of the results. With exact solutions in hand, it is possible to follow accurately the response of a particle, initially in some oscillator state, to the infinitesimal perturbation

$$-2\beta[1+\beta(2k+3)]^{1/2}x^4 + \beta^2 x^6 \qquad \beta \approx 0^+.$$

First, we note that a harmonic oscillator equation formally admits an infinite set of nodeless polynomial solutions with weight $\exp(+x^2/2)$. The corresponding energy eigenvalues are $\varepsilon_n = -(2n+1)$, $n = 0, 1, 2, \ldots$. Obviously, such solutions do not belong to the oscillator Hilbert space.

However, these are precisely the solutions of the perturbed problem that one *seems* to recover in the limit $\beta = 0$. It is tempting then to suggest that the particle responds to the switching on of the perturbation as if it were in a suitable non-normalizable state of the unperturbed problem. But this simply cannot be true. For $\beta = 0$ the particle has to be in a proper oscillator state. Indeed, there is no real problem. Consider once more the ground state function

$$\Psi_0 \sim \exp\left(\frac{\gamma x^2}{2} - \frac{\beta x^4}{4}\right)$$

for infinitesimal β. Examining the normalization integral $\int \Psi_0^2 \, dx$ for $\beta \approx 0^+$ we readily find that it is dominated by the factor $\exp(+\gamma^2/2\beta)$. Hence, it is only proper to rescale Ψ_0 appropriately by the factor $\exp(-\gamma^2/4\beta)$ prior to taking the limit. As $\beta \to 0$, the rescaled function simply disappears. The true solution for the $\beta = 0$ case should thus be sought from the corresponding Schrödinger equation. Hence, as far as the present model is concerned the root cause of the occurrence of discontinuities is *not* the loss of normalizability by the wavefunctions but their disappearance as β is made to vanish. Indeed, there is nothing surprising about the disappearance of the state function. The potential that is crucially responsible for supporting it, itself disappears abruptly in the limit.

It is also reasonably clear that the basic scenario discussed above is not the product of the specific tuning of couplings invoked here. For a fixed small β, the other coupling λ forms a quasi-discrete set since k is variable. Besides, as we have already seen, once a negative level appears, its energy is *de facto* a linear function of k. Thus we can regard k to be a real, positive and continuous parameter. The coupling λ would accordingly vary continuously. Of course, for k other than an integer the polynomial character of the solutions is lost.

In a situation like this, the failure of conventional perturbation theory is naturally to be expected. In essence, perturbation theory operates on the basis that a small perturbation implies a small modification of the state function such that, as the perturbation is gradually made to vanish, the system smoothly returns to the original state. This certainly is not the case here for any non-zero β and λ that guarantee the energetically favourable configuration of the outer wells.

As a final remark,we would like to recall our suggestion made recently [5] that quasi-exactly-solvable problems are worth far more than they have generally been credited for. The present study serves to further strengthen this view.

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